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Fractal properties in the semiclassical scattering cross section of a classically chaotic system

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Abstract. Indication is given of the way in which the semiclassical scattering cross section, as a function of the scattering angle, contains hints as to whether the corresponding classical system shows irregular scattering. In the case of classical chaos the frequency spectrum of the interference oscillations of the quantum cross section contains a smeared out image of a classical fractal structure. It becomes better and better resolved in the limit of smaller values of $\hbar$.

1. Introduction

One of the most fascinating open problems in chaos research is the understanding of the quantum mechanical behaviour of systems which show classical chaos. For bound systems some progress has been made by the investigation of level statistics and patterns of nodal lines of wavefunctions. In contrast, nothing comparable has been achieved for scattering systems. Whereas in classical scattering chaos the basic mechanism has been understood recently (see the review article by Eckhardt (1988) and references therein), only very little has been done for quantum scattering chaos so far. There is an investigation of the phase shift in the quantum mechanical scattering amplitude for a solvable model (Gutzwiller 1983). It was possible to describe the scattering phase by a Riemann ζ-function which can be considered to represent a chaotic function. An explanation of the fast fluctuations of the cross section as a function of the energy within a semiclassical approximation was given by Blümel and Smilansky (1988, 1989a, b). In this treatment some properties known from random matrix systems could be explained as quantum implications of classical scattering chaos. A treatment of resonance poles of the scattering amplitude was given within a semiclassical trajectory summation method by Cvitanović and Eckhardt (1989).

To get some further information on quantum scattering chaology, we take a system which is known to show classical chaos, treat this system semiclassically and in the limit of small $\hbar$ we find a classical fractal structure in the scattering cross section. Within a semiclassical approximation the scattering amplitude is expressed in terms of quantities given by classical scattering trajectories. So we have the occasion to trace properties of the scattering amplitude back to properties of the set of the corresponding classical trajectories. Thereby we can observe how the classical chaos enters into the quantum cross section. For simplicity we treat potential scattering in a two-dimensional position space and look at the scattering cross section as a function of the scattering angle θ for fixed incoming momentum p_{in}. We take a particular scattering potential for most of the discussions and make some remarks on the behaviour of other systems later. Our potential model is given by

$$V(x, y) = \exp[-(x+\sqrt{2})^2 - y^2] + \exp[-(x-1/\sqrt{2})^2 - (y+\sqrt{\tfrac{3}{2}})^2] + \exp[-(x-1/\sqrt{2})^2 - (y-\sqrt{\tfrac{3}{2}})^2] \quad (1)$$

where x, y are Cartesian coordinates in position space. Some contour lines of potential (1) are shown in figure 1 of Jung and Scholz (1987) (referred to hereafter as JS). This potential has a relative minimum at $E_0 \approx 0.40\ldots$ in the origin, it has three saddles at $E_S \approx 0.45\ldots$ and three maxima at $E_M \approx 1.005\ldots$. As has been demonstrated in JS, there is scattering chaos in this system if the energy lies in the interval (E_S, E_M). Its implications for the classical cross section are discussed in Jung and Pott (1989) (referred to hereafter as JP).

2. The semiclassical scattering amplitude

Incoming asymptotes of the classical scattering trajectories are labelled by the three quantities E, α, b. E is the energy, in the asymptotic region $E = (p_x^2 + p_y^2)/2$, where p_x, p_y are the momenta conjugate to x, y. α is the direction of the incoming momentum, $\alpha = \tan^{-1}(p_y/p_x)$. b is the impact parameter, $b = (xp_y - yp_x)/\sqrt{2E}$. Instead of E and α, we can use the momentum vector $\boldsymbol{p}_{\text{in}} = (p_x, p_y)$ equally well. The scattering angle θ is the difference between the direction of the outgoing and the direction of the incoming momentum. The semiclassical approximation of the scattering amplitude for the scattering angle $\theta = \bar{\theta}$ is given by

$$f(\bar{\theta}) = \sum_j \sqrt{c_j} \exp\{\mathrm{i}[S_j(\bar{\theta})/\hbar - \mu_j\pi/2]\} \quad (2)$$

(see, e.g., Miller 1975). The summation runs over all classical scattering trajectories, that come in with momentum $\boldsymbol{p}_{\text{in}}$ and have a scattering angle $\bar{\theta}$. Let b be the initial impact parameter and $\theta(b)$ the scattering angle as a function of b for fixed $\boldsymbol{p}_{\text{in}}$. Then the sum in (2) runs over all b values $b_j(\bar{\theta})$ which lead to $\theta(b_j) = \bar{\theta}$. $c_j = |(\mathrm{d}\theta/\mathrm{d}b)(b_j(\bar{\theta}))|^{-1}$ is the contribution of trajectory j to the classical cross section. S_j is the reduced action of trajectory j,

$$S_j = -\int \boldsymbol{q}\, \mathrm{d}\boldsymbol{p} = -\int (x\, \mathrm{d}p_x + y\, \mathrm{d}p_y).$$

In polar coordinates r, φ and their conjugate momenta p_r, L we find

$$S_j = -\int r\, \mathrm{d}p_r + \int L\, \mathrm{d}\varphi.$$

The line integral is taken along the trajectory j, and the integral is independent of the initial and final point on this trajectory as long as both of these points are located sufficiently far away from the potential region. μ_j is the Maslov index of trajectory j, i.e. the number of caustics met by trajectory j. Equation (2) is the leading asymptotic approximation of the scattering amplitude for $\hbar \to 0$ as long as $\bar{\theta}$ stays away from rainbow angles, at which some $(\mathrm{d}\theta/\mathrm{d}b)(b_j)$ becomes zero. There the approximation by exponential functions has to be replaced by an appropriate uniform approximation. In this paper we shall consider the scattering amplitude in angle intervals without rainbow singularities only.

For system (1) there is a hyperbolic invariant set Λ in the classical phase space. The stable manifolds of these localised orbits reach out into the asymptotic region and intersect the plane of incoming asymptotes in a fractal subset (see figure 9 in JS). Let us assume, that $\boldsymbol{p}_{\text{in}}$ is fixed at a value such that this fractal set is met when we scan b. Then the b axis is cut into an infinite number of intervals of continuity, in which $\theta(b)$ is a smooth function. In between these intervals there remains a Cantor set at which $\theta(b)$ jumps (see JS for more details). Such a behaviour is typical for all chaotic scattering systems, which have been analysed so far. Let us look at a range of θ values, in which unstable manifolds of Λ are present. Under these conditions there is an infinite number of b values b_j, which lead to a given θ value.

This can be understood as follows. We take the b axis for the given $\boldsymbol{p}_{\text{in}}$, transport it through the classical phase space by the flux, thereby construct the Lagrangian submanifold $\mathscr{L}(\boldsymbol{p}_{\text{in}})$ belonging to the initial condition $\boldsymbol{p}_{\text{in}}$ and finally take the intersection between $\mathscr{L}(\boldsymbol{p}_{\text{in}})$ and the θ/L plane of outgoing asymptotes, where L is the angular momentum of the outgoing trajectory. Thereby each interval of continuity along the b axis is mapped on a smooth curve in the θ/L plane. When b goes to the boundary of the interval, the image curve spirals towards a boundary line, which is the intersection of the θ/L plane with one branch of the unstable manifold of the periodic orbit oscillating over the saddle of the potential, through which the trajectories leave the potential interior. This behaviour is illustrated in figure 1. $\boldsymbol{p}_{\text{in}}$ is fixed at $(-\sqrt{1.2}, 0)$ and b is scanned in three intervals of continuity (R, LL, LR in the notation of JS). In the outgoing θ/L plane one plots those three curves which are reached by trajectories starting in these three intervals. The curves to the infinite number of other intervals which go through the same saddle give similar spirals, converging towards the same boundary line. This infinity of spirals is arranged in such a way that their accumulation points form a fractal pattern, coinciding with the intersection of the unstable manifolds

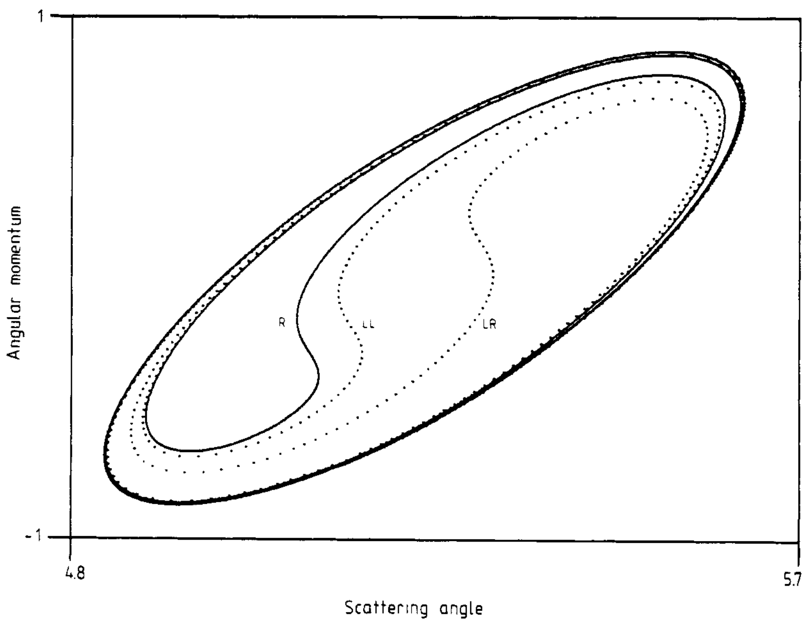

Figure 1. Image of the intervals R, LL and LR in the θ/L plane of outgoing asymptotes for $\boldsymbol{p}_{\text{in}} = (-\sqrt{1.2}, 0)$.

of Λ with the θ/L plane (for more explanations see JP). By $D(\bar{\theta})$ we denote the set of L values which lie in the intersection between $\mathscr{L}(\boldsymbol{p}_{\text{in}})$ and the line $\theta=\bar{\theta}$ in the outgoing θ/L plane. The accumulation points of $D(\bar{\theta})$ form a fractal set along the line $\theta=\bar{\theta}$. The trajectories corresponding to points of $D(\bar{\theta})$ are the ones which contribute to the sum in (2).

System (1) has the property that for E not too close to the saddle energy $E_S \approx 0.45\ldots$ the sum in (2) is absolutely convergent, as long as the angle $\bar{\theta}$ is not at a rainbow singularity. First we consider the contributions coming from one particular interval of continuity. If b_k is a contributing b value and b_n is the second contributing b value closer to the boundary of the interval, then $c_n \approx c_k/\mu$, where μ is the eigenvalue of the unstable periodic saddle trajectory. For $E=0.6$ we find $\mu \approx 107$. Therefore the contributions from one particular interval can be estimated by a geometric series. To sum over all intervals in a second step, we sort the intervals into groups of various generation given by the length of their signature (see JS) and obtain 2^N intervals of generation N. There exists a number $\nu > 4$, such that going from any interval of generation N to a neighbouring interval of generation $N+1$ the ratio of the length of these two intervals is greater than ν. There is a one-to-one correspondence between the various contributions c_k of any pair of intervals (see JP). The ratio of the strengths of the corresponding contributions is proportional to the ratio of the lengths of the respective intervals. Therefore the ratio between the sum of all contributions of the intervals of generation $N+1$ and all intervals of generation N in the sum (2) is less than $2/\sqrt{\nu}<1$. Accordingly, the sum over all intervals can also be estimated by a geometric series, showing the convergence of the total sum. For most chaotic scattering systems there can be energy intervals for which the semiclassical sum in (2) is not absolutely convergent. In these cases an appropriate resummation scheme has to be applied. It might be constructed along the pattern of rearrangement of the semiclassical series mentioned in Cvitanović and Eckhardt (1989).

When it converges absolutely, then the sum in (2) can be cut off in a numerical treatment. We fix some numerical error boundary and estimate the number of contributions we need in order to stay within this error boundary for the value of the amplitude.

3. Extraction of a classical fractal set out of the semiclassical cross section

Now we look for fingerprints of the classical chaos in the semiclassical cross section. We pick out a θ interval $I=[\bar{\theta}, \bar{\theta}+\Delta\theta]$ away from all classical rainbow singularities. Then $(\mathrm{d}\theta/\mathrm{d}b)(b_j) \neq 0$ for all j and all $\theta \in I$, the number of solutions of $b(\theta)$ does not change inside I and c_j varies only slowly inside I along any branch of $\mathscr{L}(\boldsymbol{p}_{\text{in}})$ and we approximate

$$c_j(\theta) = \left|\frac{\mathrm{d}\theta}{\mathrm{d}b}(b_j(\theta))\right|^{-1} = c_j(\bar{\theta}) \tag{3}$$

$S(\theta)$ is expanded up to first order around $\bar{\theta}$

$$S_j(\theta) = S_j(\bar{\theta}) + (\theta-\bar{\theta})\frac{\mathrm{d}S_j}{\mathrm{d}\theta}(\bar{\theta}) \tag{4}$$

where $(\mathrm{d}S_j/\mathrm{d}\theta)(\bar{\theta}) = L_j(\bar{\theta})$ is the outgoing angular momentum. Using the notation

$$\varphi_j = [S_j(\bar{\theta}) - \bar{\theta}L_j(\bar{\theta})]/\hbar - \mu_j\pi/2 \tag{5}$$

we obtain

$$f(\theta)=\sum_j \sqrt{c_j}\,\exp(\mathrm{i}\varphi_j)\exp(\mathrm{i}\theta L_j)\hbar) \tag{6}$$

f is the Fourier transform of

$$F(L)=\sum_j \sqrt{c_j}\,\exp(\mathrm{i}\varphi_j)\delta(L-L_j).$$

The support of F is the classical fractal set $D(\bar{\theta})$. For the cross section we obtain

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\theta}(\theta)=|f(\theta)|^2=\sum_j c_j+\sum_{k<j}\sqrt{c_jc_k}\,2\cos(\varphi_k-\varphi_j+\theta(L_k-L_j)/\hbar). \tag{7}$$

The frequencies occurring in the interference terms are given by $(L_k-L_j)/\hbar$; they also form a fractal distribution.

Let us illustrate some of these quanties for system (1). We choose $\bar{\theta}=5.4$, which is far away from all classical rainbow singularities (see figure 1 and figures in JP). Figure 2(a) shows the position of the most important values of L_k-L_j and the values of the corresponding weights $\sqrt{c_kc_j}$. The horizontal axis gives L_k-L_j on a linear scale. The vertical axis gives $\ln\sqrt{c_kc_j}$. Note that there is an isolated branch of $\mathscr{L}(\boldsymbol{p}_{\mathrm{in}})$ at

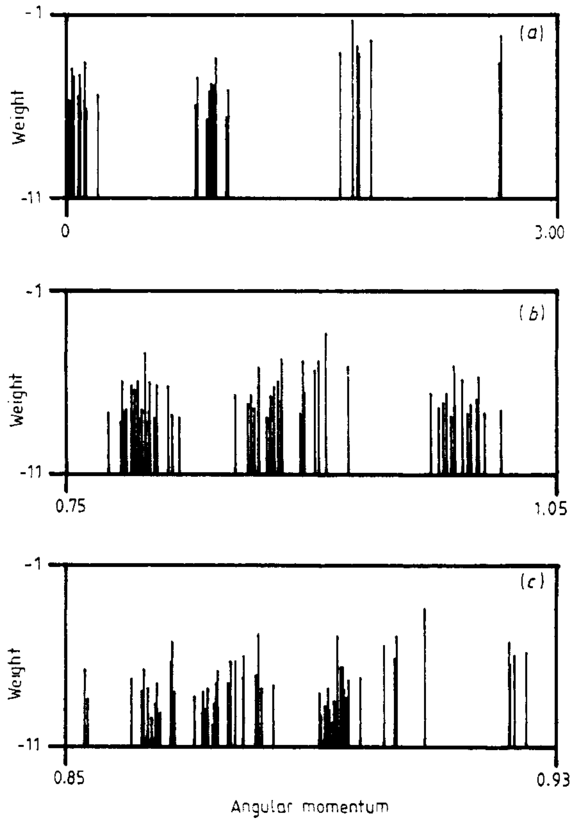

Figure 2. Plot of $\ln\sqrt{c_kc_j}$ (vertical axis) as a function of L_k-L_j (horizontal axis). The 9, 17 and 32 most important branches of $\mathscr{L}(\boldsymbol{p}_{\mathrm{in}})$ have been taken into account in parts (a), (b) and (c) respectively. $\boldsymbol{p}_{\mathrm{in}}=(-\sqrt{1.2},0)$.

$L \approx 2.6$ which is outside the frame of figure 1. This branch comes from trajectories which pass the potential on the outside and do not enter the potential interior through one of the saddles. This branch has the strongest c_j and is involved in all high values of $L_k - L_j$ which lie around 2 and 2.8.

Figures 2(*b*) and 2(*c*) are magnifications of figure 2(*a*) which illustrate the fractal character of the distribution of L values. In figure 2(*a*) the 9 most important branches of $\mathscr{L}(\boldsymbol{p}_{\text{in}})$ have been taken into account; in figure 2(*b*), 17 and in figure 2(*c*) 32 branches contribute.

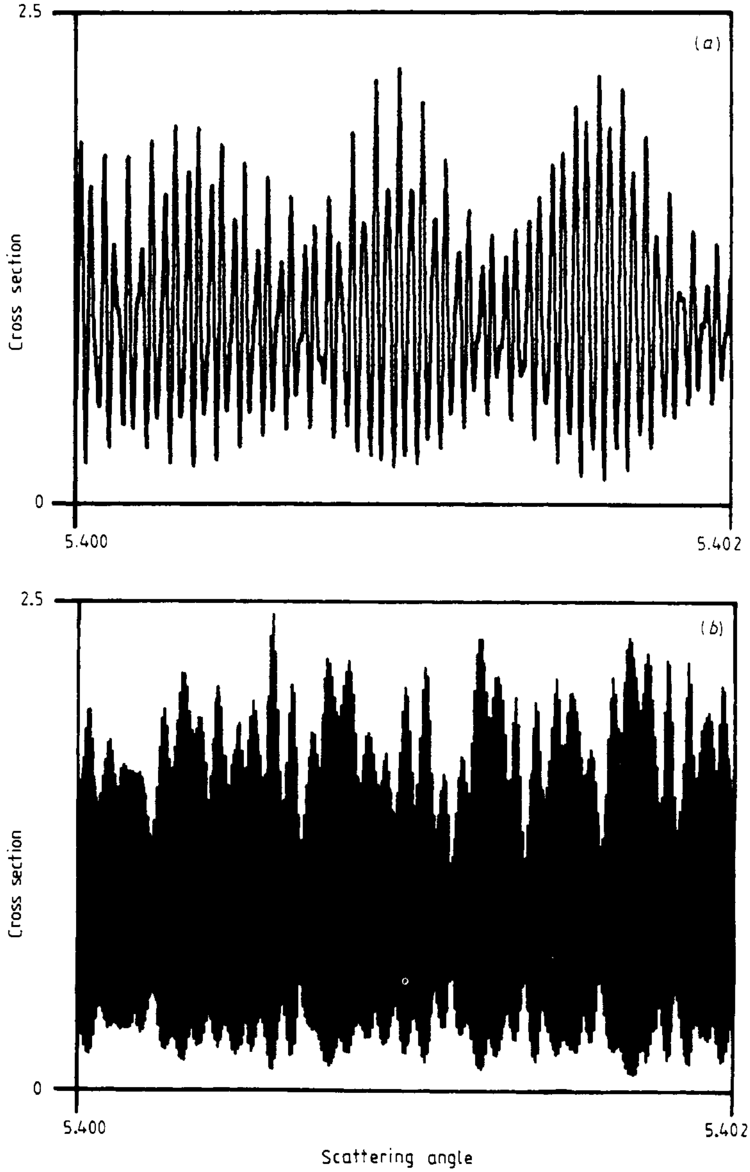

Figure 3. Plot of the semiclassical cross section as a function of the scattering angle. The 32 most important branches of $\mathscr{L}(\boldsymbol{p}_{\text{in}})$ have been taken into account. $\boldsymbol{p}_{\text{in}} = (-\sqrt{1.2}, 0)$. $\hbar = 10^{-5}$ in part (*a*) and $\hbar = 10^{-6}$ in part (*b*).

Figure 3 shows $(d\sigma/d\theta)(\theta)$ in the interval [5.4, 5.402] where in sum (2) the 32 most important contributions have been taken. The cutoff of sum (2) after 32 terms leads to a relative error in the cross section of less than 4%. For $\hbar$ the value $\hbar = 0.000\,01$ has been chosen in figure 3(a) and $\hbar = 0.000\,001$ in figure 3(b). In figure 3(b) the fast fluctuations are not resolved, they are of the same qualitative structure as in figure 3(a) only compressed by a factor of 10. In figure 3(b) we see the fluctuations of the envelope coming from the small values of $L_k - L_j$.

Finally we suppose that $(d\sigma/d\theta)(\theta)$ is given inside I and we try to reconstruct $D(\bar{\theta})$ from these data. By taking a Fourier transform of $d\sigma/d\theta$ over the θ interval I of length $\Delta\theta$, we do not get infinitely sharp values of L; instead, for each contributing L value we obtain a broadened peak. The width of the peaks comes from two sources. First, the finite length $\Delta\theta$ of the interval I causes a width $\Delta_1 = 2\pi\hbar/\Delta\theta$. Second, the nonlinear terms omitted in expansion (4) create a broadening. Expansion (4) only makes sense if L_j is sufficiently constant inside the interval I along each branch of $\mathscr{L}(\boldsymbol{p}_{\text{in}})$. Choose δ such that

$$\left|\frac{d^2 S_j(\theta)}{d\theta^2}\right| = \left|\frac{dL_j}{d\theta}\right| < \delta \qquad \text{for all } j \text{ and all } \theta \in I.$$

Then $|L_j(\theta) - L_j(\bar{\theta})| < \delta\,\Delta\theta = \Delta_2$. And the broadening of the L values due to the nonlinearity of $S_j(\theta)$ is at most Δ_2. For given values of $\hbar$ and $\Delta\theta$ we can expect to resolve $D(\bar{\theta})$ to an accuracy of $\Delta = \Delta_1 + \Delta_2$. When we make $\hbar$ smaller and smaller we let $\Delta\theta$ decrease like $\hbar^{1/2}$. Then Δ_1 and Δ_2 both decrease like $\hbar^{1/2}$. By repeating the procedure for various values of $\bar{\theta}$ we can reconstruct the classical Lagrangian submanifold $\mathscr{L}(\boldsymbol{p}_{\text{in}})$ with an accuracy $\Delta \propto \hbar^{1/2}$. In total, we have pulled a classical fractal set out of a quantum mechanical observable quantity, at least approximately in the limit of small $\hbar$. What would be the procedure for realising this idea for an actual scattering system? We take a system where an enormous number of L values contribute to a given θ value and analyse the weight of the various L values in $d\sigma/d\theta$. We try out various lengths $\Delta\theta$ of the θ interval which we take for the Fourier transform, in order to find the $\Delta\theta$ value which gives the optimal resolution. For $\Delta\theta$ too small we obtain a large width Δ_1 and for $\Delta\theta$ too large Δ_2 becomes too large. For the best value of $\Delta\theta$ we plot the weights of the various L values against L and look for a fractal clustering which resembles the structure of figure 2. This method opens the possibility of finding hints for chaos in the quantum mechanical cross section as a function of the scattering angle.

4. Discussion

We have observed that in the limit of small $\hbar$ the spectrum of frequencies of the interference oscillations of the semiclassical cross section simulates a fractal structure well known from the classical system. We stress that we let $\hbar$ tend to small values but cannot take the value $\hbar = 0$ itself. $\hbar = 0$ is an essential singularity of quantum mechanics, and in the classical cross section all interference oscillations are completely absent. Accordingly, we can only look at the behaviour of the cross section for smaller and smaller values of $\hbar$ and see more and more levels of the classical fractal structure being resolved. We never reach a simultaneous resolution of the classical fractal structure on its infinite number of levels of scale.

The effects shown for system (1) do not exist for every scattering system. The topological chaos in the classical phase space is caused by homoclinic and heteroclinic intersections between the stable and unstable manifolds of unstable periodic orbits running back and forth over the saddles of the potential. Accordingly, we expect chaos of similar structure only for those other scattering systems where the potential also has saddles and unstable periodic orbits with homoclinic and/or heteroclinic connections. Under these conditions and for $\boldsymbol{p}_{\rm in}$ taken from an appropriate range of values, $\mathscr{L}(\boldsymbol{p}_{\rm in})$ is a fractal arrangement of branches in the outgoing asymptotic region. For more general chaotic scattering systems we cannot expect to find a binary organisation in the fractal sets as we did in model system (1) (see JS and JP). The fractal set of discontinuities of $\theta(b)$ along the b axis, the distribution of rainbows along the θ axis and $\mathscr{L}(\boldsymbol{p}_{\rm in})$ may all have a more complicated organisation. However, in all chaotic scattering systems the outgoing L values contributing to a particular value of θ form a fractal set as long as the initial condition $\boldsymbol{p}_{\rm in}$ lies in the chaotic region. Under these conditions the interference oscillations of the semiclassical cross section contain a smeared out image of the fractal structure of the classical Lagrangian submanifold $\mathscr{L}(\boldsymbol{p}_{\rm in})$.

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